Satellite Operators on Knot Concordance

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Topology in dimension n : $n = \underbrace{1 \;,\; 2 \;,\; 3}_{\text{geometric techniques}}, 4, \underbrace{5,\; 6,\; 7, \ldots\ldots}_{\text{algebraic techniques}}$

Two motivating questions

(1) When does algebra determine topology?

Thm [Freedman]

X^4 closed, simply connected, $H_*(X) \cong H_*(S^4) \Rightarrow X \underset{TOP}{\cong} S^4$

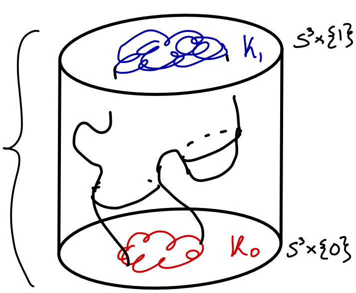
(2) When does topology determine smooth structure?

Thm [Akbulut-Ruberman]

$\exists$ compact contractible 4-manifolds w/ distinct smooth structures

Model setting: Knot Concordance

$S^3 \times I$



K_0 and K_1 are **$\star$-concordant** ($K_0 \sim_\star K_1$) if they cobound a $\star$-embedded annulus in $S^3 \times I$

$\star \in \{\text{smooth, top(ologically locally flat)}\}$

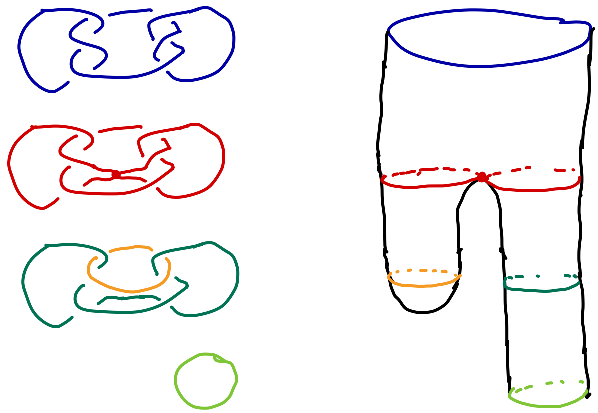
Example

$S^3\times\{1\}$

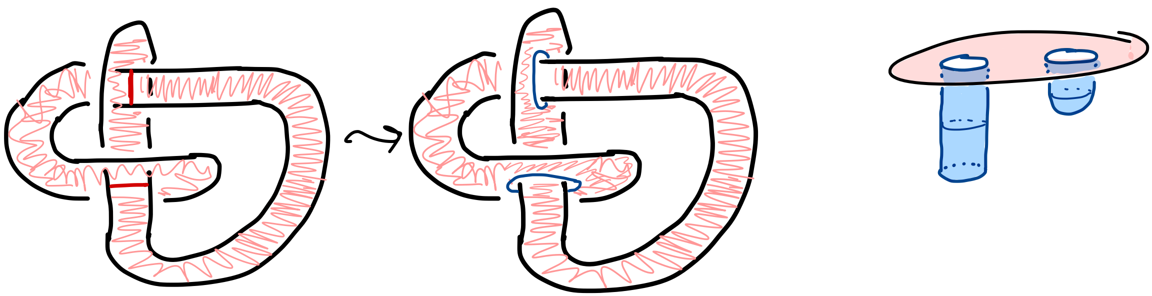
$S^3\times\{2/3\}$

$S^3\times\{1/3\}$

$S^3\times\{0\}$



Example 2



Why might one care?

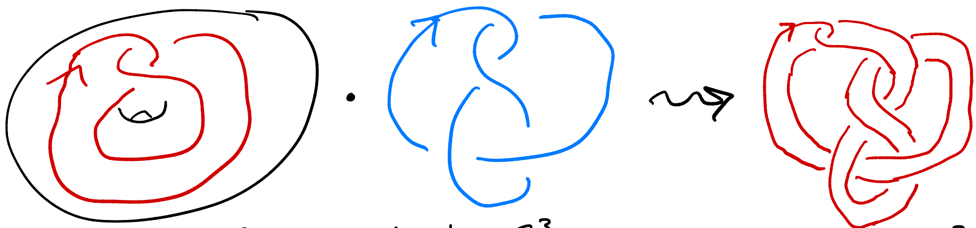
$\left(\exists\, U \underset{TOP}{\sim} K \underset{SM}{\not\sim} U\right) \Longrightarrow \left(\exists \text{ multiple distinct smooth structures on } \mathbb{R}^4\right)$

Theorem (Fox-Milnor) $\mathcal{C}_* := \{\text{knots in } S^3\}/\sim_*$ is an abelian group (wrt operation induced by #)

Known: $\mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty \subseteq \mathcal{C}_*$

Unknown: $\mathbb{Q} \subseteq \mathcal{C}_*$? $\mathbb{Q}/\mathbb{Z} \subseteq \mathcal{C}_*$?

Philosophy: Understand $\mathcal{C}_*$ via satellite operators



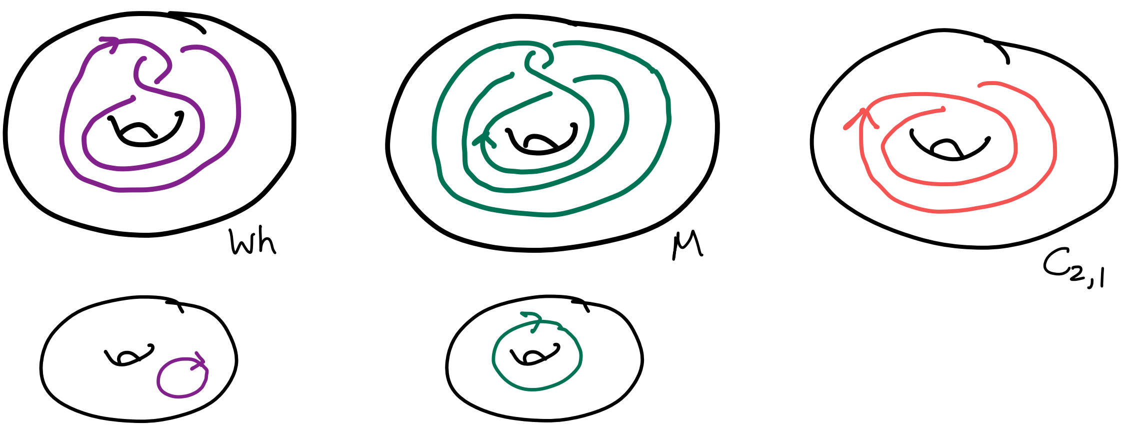
$P: S^1 \hookrightarrow S^1 \times D^2$ $\quad\cdot\quad$ $K: S^1 \hookrightarrow S^3$ $\quad\rightsquigarrow\quad$ $P(K): S^1 \hookrightarrow S^3$

Useful Fact.

Every pattern P induces $P: \mathcal{C}_* \to \mathcal{C}_*$.

given $K_0 \sim_* K_1$, "satellite the concordance" to see $P(K_0) \sim_* P(K_1)$.

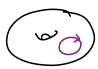
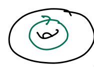
Examples



Wh $\quad$ M $\quad$ $C_{2,1}$

Key attribute: algebraic winding # $\omega_P \in \mathbb{Z}$

Question 1 What are the set-theoretic properties of $P: \mathcal{C}_* \to \mathcal{C}_*$

Winding Number:	0	± 1	$\omega \notin \{-1, 0, 1\}$
Surjective?	Never	Sometimes!  + (1) Not always. (sm) (2)	Never
Injective?	Not always  + (3) Ever?	Sometimes!  + (1) Always?	???

(1) M.-Piccirillo, 2018

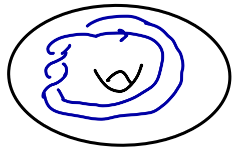
(2) A. Levine, 2016

(3) M., 2019

Question 2 How do satellite operators interact with additional structure on $\mathcal{C}_*$?

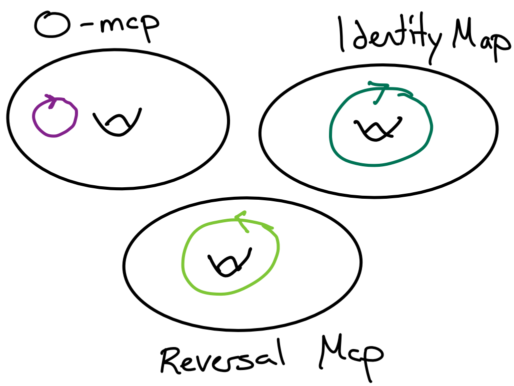
More precisely: when is $P: \mathcal{C}_* \to \mathcal{C}_*$ a group homomorphism.

(1) Not always: $P(U)$ must be concordant to U



} Not a hom.

(2) Sometimes!



(3) Conjecture [Hedden 2016]:
The only satellite-induced homomorphisms of $\mathcal{C}_*$ are $[K] \to [K]$, $[K] \to [K^r]$, and $[K] \to [U]$.

Technical Challenge:

$P(U)$ slice $\Rightarrow$ P induces a hom of "algebraic concordance"

Some results:

(1) These are not homomorphisms of $\mathcal{C}_{sm}$.

[Recent work w/ Lidman and Pinzón-Caicedo unifies + generalizes (most of) these examples.]

Moral: We can obstruct given examples from inducing homs, but thus far no general strategy.

Def'n

$$g_4^*(K) := \min\{g(F) : F \overset{*}{\hookrightarrow} B^4, \partial F = K\}$$

(generalizes $g_3(K) = \min\{g(F) : F \hookrightarrow S^3, \partial F = K\}$.)

<u>Thm</u> [Schubert, 1952]

P a pattern. There exists $g_P \in \mathbb{N}_{\geq 0}$ for every nontrivial K,

$$g_3(P(K)) = g_P + |w_P|\, g_3(K)$$

<u>Corollaries</u>

(a) $g_3(C_{n,1}(T_{2,3})) = n$

(b) $\displaystyle\lim_{k\to\infty} \frac{g_3(P(T_{2,2k+1}))}{g_3(T_{2,2k+1})} = |w_P|$ for all P

What happens w/ smooth 4-genus?

(a) $g_4^{sm}(C_{n,1}(T_{2,3})) = n$

(b) $\displaystyle\lim_{k\to\infty} \frac{g_4^{sm}(P(T_{2,2k+1}))}{g_4^{sm}(T_{2,2k+1})} = |w_P|$ for all P

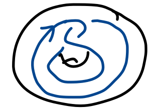
What happens w/ topological 4-genus?

<u>Thm</u> [Feller-M-Pinzón-Caicedo, '19]

(a) $g_4^{top}(C_{n,1}(T_{2,3})) = \underline{1}$.

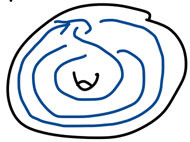
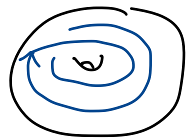
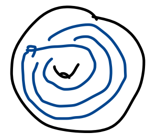
(b) $\displaystyle\lim_{k\to\infty} \frac{g_4^{top}(P(T_{2,2k+1}))}{g_4^{top}(T_{2,2k+1})} = \begin{cases} \underline{1} & w_P \neq 0 \\ 0 & w_P = 0 \end{cases}$

Some outstanding problems



(1) Prove that $Wh(K) \underset{sm}{\sim} U \iff K \underset{sm}{\sim} U$

[More generally, prove there are injective-but-not-surjective patterns:

   .....]

(2) Determine whether every winding # 1 pattern acts by connected sum on top. concordance.

[• Related to "Akbulut-Kirby" conjecture.
• If yes, "homotopy slice-ribbon" is false.]

(3) Is it true that for all P, K

$$g_4^{top}(P(K)) \leq g_4^{top}(P(U)) + g_4^{top}(K) \ ?$$

[Note:
For any notion of knot genus g,

$$g(P(K)) \leq g_P + |w_P|\, g(K).$$

For $g \in \{g_3, g_4^{sm}\}$ there are many examples of equality.]